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A model of the Calogero type in the *D*-dimensional space

A Bachkhaznadji[1], **M Lassaut**[2] **and R J Lombard**[2]

[1] Laboratoire de Physique Théorique, Département de Physique, Université Mentouri, Constantine, Algeria

[2] Groupe de Physique Théorique, Institut de Physique Nucléaire, 91406 Orsay Cedex, France

E-mail: ma.bachkhaznadji@yahoo.fr, lassaut@ipno.in2p3.fr and lombard@ipno.in2p3.fr

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Abstract

In a previous paper, we have proposed a new integrable Hamiltonian describing two interacting particles in a harmonic mean field in $D = 1$ dimensional space. Here, we generalize this Hamiltonian to the $D \geqslant 2$ dimensional space. We show that the system is exactly solvable for a certain domain of the interaction constant, and that the size of this domain increases with D. We also show this D-dimensional Hamiltonian to be supersymmetric and shape invariant, very much as in the $D = 1$ case.

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1. Introduction

Most of the quantum N-particle systems studied so far concern N particles interacting on the line, as the Calogero model [1], or on the circle as the Sutherland model [2]. A survey of such quantum integrable systems was done by Olshanetsky and Perelomov [3] some years ago. They classified the systems with respect to Lie algebras. Point interactions have also been considered, still in the $D = 1$ dimensional space [4, 5].

In a recent paper, we have studied a new integrable model of the 'Calogero type'. It describes two interacting particles in a harmonic field, this last being thought as generated by an infinitely heavy third particle [6]. The work was done in the $D = 1$ dimensional space. The exact solutions of the corresponding Hamiltonian were found. The model was shown to be both supersymmetric and shape invariant. A generalized version of this model has been analysed by Meljanac *et al* [7]. These authors pointed out the underlying $SU(1, 1)$ symmetry.

The purpose of the present work is to show this model to be exactly solvable in the D-dimensional space. Furthermore, the supersymmetry and shape invariance properties are valid whatever D is.

The paper is organized as follows. In section 2, we study the case for $D = 3$. We determine the domain of the coupling constant for which the system is exactly solvable. Because of its particularities linked to the behaviour at short distances, the case $D = 2$ is examined in section 3. In section 4, the problem is generalized to the D-dimensional space. In section 5, we show the model to be supersymmetric. The corresponding potentials for the angular and radial equations belong to the class of the shape invariant potentials for any D. The $SU(1, 1)$ symmetry is discussed in section 6. Conclusions are drawn in section 7.

2. The model in $D = 3$ dimensions

We consider two interacting particles within a harmonic mean field in the $D = 3$ dimensional space. It is the straightforward generalization of the 1D model studied in [6]. The Hamiltonian is given by

$$H(\vec{r}_1, \vec{r}_2) = -\Delta_1 - \Delta_2 + \omega^2\left(\vec{r}_1^{\,2} + \vec{r}_2^{\,2}\right) + 4\lambda \frac{2\vec{r}_1.\vec{r}_2}{\left(\vec{r}_1^{\,2} + \vec{r}_2^{\,2}\right)|\vec{r}_1 - \vec{r}_2|^2}, \tag{1}$$

where $\Delta_{1,2}$ are the 3D Laplacian of particles 1 and 2, respectively. Here, use is made of units $2m = \hbar = 1$. Note that this Hamiltonian is invariant in the permutation of particles 1 and 2. It is not translationally invariant, but this has no consequence, since the origin of the mean field is given by the position of the infinitely heavy particle.

Introducing the centre of mass and relative variables

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{\sqrt{2}} \qquad \vec{s} = \frac{\vec{r}_1 - \vec{r}_2}{\sqrt{2}}, \tag{2}$$

we obtain the Schrödinger equation

$$\left(-\Delta_{\vec{R}} - \Delta_{\vec{s}} + \omega^2(R^2 + s^2) + 2\lambda \frac{R^2 - s^2}{(R^2 + s^2)s^2}\right)\psi(\vec{R}, \vec{s}) = E\psi(\vec{R}, \vec{s}). \tag{3}$$

Since the potential does not depend on the angle between $\vec{R}$ and $\vec{s}$, the separation of angular and radial variables

$$\psi(\vec{R}, \vec{s}) = \frac{\psi^{(\ell,\ell')}(R, s)}{Rs} Y_{\ell,m}(\Omega_R) Y_{\ell',m'}(\Omega_s) \tag{4}$$

leads to

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial s^2} - \frac{\ell(\ell+1)}{R^2} - \frac{\ell'(\ell'+1)}{s^2} - \omega^2(R^2+s^2) - 2\lambda \frac{R^2 - s^2}{(R^2+s^2)s^2} + E\right)\psi^{(\ell,\ell')}(R, s) = 0. \tag{5}$$

It is convenient to introduce polar coordinates

$$R = r\cos\theta, \qquad s = r\sin\theta. \tag{6}$$

They range over $0 \leqslant r < \infty$ and $\theta \in [0, \pi/2]$ due to the positivity of R, s.

Setting $\psi^{(\ell,\ell')}(R, s) = \Phi(r, \theta)$ for fixed ℓ and ℓ', equation (5) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\ell(\ell+1)}{r^2(\cos\theta)^2} - \frac{\ell'(\ell'+1)}{r^2(\sin\theta)^2} - \omega^2 r^2 - \frac{2\lambda}{r^2}\frac{(\cos\theta)^2 - (\sin\theta)^2}{(\sin\theta)^2} + E\right)\Phi(r, \theta) = 0. \tag{7}$$

We look for eigenenergies $E_{k,n}$ and eigensolutions $\Phi_{k,n}$ of the above Hamiltonian, where k and n label the radial and angular states, respectively.

Assuming

$$\Phi_{k,n}(r,\theta) = u_k(r)\phi_n(\theta), \tag{8}$$

equation (7) separates in two differential equations:

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\ell(\ell+1)}{(\cos\theta)^2} - \frac{\ell'(\ell'+1)}{(\sin\theta)^2} - 2\lambda\frac{(\cos\theta)^2-(\sin\theta)^2}{(\sin\theta)^2} + C_n\right)\phi_n(\theta) = 0 \tag{9}$$

and

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \omega^2 r^2 + E_{k,n} - \frac{C_n}{r^2}\right]u_k(r) = 0. \tag{10}$$

A first remark concerns the domain of validity of (9). Its potential term has a periodicity of π. The singularities occur at $\theta = k\pi/2$. Here $\theta \in [0, \pi/2]$. The equation is solved on $[0, \pi/2]$. It is recast as

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\ell(\ell+1)}{(\cos\theta)^2} - \frac{\ell'(\ell'+1)+2\lambda}{(\sin\theta)^2} + 4\lambda + C_n\right)\phi_n(\theta) = 0. \tag{11}$$

Since $(\sin\theta)^2$ behaves like θ^2 in the vicinity of 0 the singularity is analogous to that of a centrifugal barrier. It can be treated if and only if $1/4 + 2\lambda + \ell'(\ell'+1) > 0$ [8], otherwise the operator has several self-adjoint extensions. To ensure that $1/4 + 2\lambda + \ell'(\ell'+1) > 0$ is valid for every ℓ', we consider the values of λ such that $1/4 + 2\lambda > 0$. As far as the centrifugal barrier $\ell(\ell+1)/(\cos\theta)^2$ is concerned, it behaves like $\ell(\ell+1)/(\theta-\pi/2)^2$ in the vicinity of $\pi/2$ and does not raise difficulties.

Let us introduce $\tilde{\ell}$ defined by

$$\tilde{\ell}(\tilde{\ell}+1) = \ell'(\ell'+1) + 2\lambda \tag{12}$$

or equivalently

$$\tilde{\ell} = -1/2 + \sqrt{(\ell'+1/2)^2 + 2\lambda}. \tag{13}$$

Here we consider the positive root of (12) in order to recover $\tilde{\ell} = \ell'$ when $\lambda = 0$. Equation (11) becomes

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\ell(\ell+1)}{(\cos\theta)^2} - \frac{\tilde{\ell}(\tilde{\ell}+1)}{(\sin\theta)^2} + 4\lambda + C_n\right)\phi_n(\theta) = 0. \tag{14}$$

A similar equation has been studied in [3] and [9]. It resembles the Pöschl–Teller equation. The solutions on the interval $[0, \pi/2]$ with Dirichlet conditions at the boundaries are expressed in terms of orthogonal polynomials. This is achieved by the ansatz [9]

$$\phi_n(\theta) = (\sin\theta)^{\tilde{\ell}+1}(\cos\theta)^{\ell+1}g_n(y) \qquad y = \cos 2\theta. \tag{15}$$

Definition (13) assures the positivity of the $\sin\theta$ exponent $\forall\ell'$. The ansatz (15) transforms (14) to

$$\left[(1-y^2)\frac{\partial^2}{\partial y^2} + (\ell - \tilde{\ell} - (3+\ell+\tilde{\ell})y\frac{\partial}{\partial y} + \frac{C_n}{4} + \lambda - \frac{(\ell+\tilde{\ell}+2)^2}{4}\right]g_n(y) = 0 \tag{16}$$

with (see equation (13))

$$\lambda = \frac{(\tilde{\ell}-\ell')(\tilde{\ell}+\ell'+1)}{2}. \tag{17}$$

The general solutions are the Jacobi polynomials $P_n(\tilde{\ell}+1/2, \ell+1/2, y)$ [10], provided that C_n fulfils the condition

$$C_n = -4\lambda + (\ell+\tilde{\ell}+2)^2 + 4n(n+\ell+\tilde{\ell}+2). \tag{18}$$

Accordingly, the general solution of (11) is of the form

$$\phi_n(\theta) = (\sin\theta)^{\tilde{\ell}+1}(\cos\theta)^{\ell+1} P_n^{\tilde{\ell}+1/2,\ell+1/2}(\cos 2\theta). \tag{19}$$

Turning to the radial equation, using $u_k(r) = v_k(r)/\sqrt{r}$ leaves us with the reduced radial equation

$$\left[\frac{\partial^2}{\partial r^2} - \omega^2 r^2 - \frac{C_n - 1/4}{r^2} + E_{k,n}\right] v_k(r) = 0. \tag{20}$$

The constant C_n has to be strictly positive in order to ensure this Hamiltonian to be a self-adjoint operator. For $C_n = 0$, it has several self-adjoint extensions. This can be avoided by setting $C_n > 0$ to be satisfied for every non-negative integer n. Taking into account that $n(n + \ell + \tilde{\ell} + 2)$ has zero for its lower limit, this yields the condition

$$-4\lambda + (\ell + \tilde{\ell} + 2)^2 > 0. \tag{21}$$

Recalling the definition of $\tilde{\ell}$ (equation (13)), and that ℓ is a positive integer or zero, the minimal value of $\tilde{\ell}$, which we denote $\tilde{\ell}_c$, is given by

$$\tilde{\ell}_c = -\frac{1}{2} + \sqrt{2\lambda + \frac{1}{4}}. \tag{22}$$

In (21) the squared term has a lowest value given by $(\tilde{\ell}_c + 2)^2$. Accordingly, the condition $C_n > 0\,\forall\, n, \ell, \ell'$ is ensured if

$$-4\lambda + (\tilde{\ell}_c + 2)^2 > 0. \tag{23}$$

This is verified if

$$-\frac{1}{8} \leqslant \lambda < \frac{7 + 3\sqrt{5}}{2}. \tag{24}$$

It is convenient to introduce the auxiliary quantity ℓ_F, defined by

$$C_n = \ell_F^2 \qquad \ell_F = \sqrt{(\ell + \tilde{\ell} + 2n + 2)^2 - 4\lambda}. \tag{25}$$

It allows us to write the square integrable solutions of equation (20) as

$$v_k(r) = r^{\ell_F + 1/2} \exp(-\omega r^2/2) f_k(\omega r^2). \tag{26}$$

It is obvious that ℓ_F has to be the positive root of C_n to ensure the squared integrability of the solutions, and the correct behaviour of $u_k(r)$ at the origin.

Inserting this ansatz in the reduced radial equation, we obtain the differential equation for f_k:

$$z f_k''(z) + (\ell_F + 1 - z) f_k'(z) + \left(\frac{E_{k,n}}{4\omega} - \frac{\ell_F}{2} - \frac{1}{2}\right) f_k(z) = 0, \tag{27}$$

where $z = \omega r^2$ and the prime denotes the derivative with respect to z. Equation (27) is nothing but the Laguerre polynomials's differential equation with solutions

$$f_k(z) = L_k^{(\ell_F)}(z), \qquad k = 0, 1, 2, \ldots. \tag{28}$$

The corresponding spectrum of the radial equation is given by

$$E_{k,n} = 2\omega(2k + \ell_F + 1). \tag{29}$$

The complete expression of the eigenvalues reads

$$E_{k,n}(\ell, \tilde{\ell}) = 2\omega\big(2k + 1 + \sqrt{(\ell + \tilde{\ell} + 2n + 2)^2 - 4\lambda}\big). \tag{30}$$

The eigenfunctions are given by

$$\Phi_{k,n}(r,\theta) = r^{\ell_F}\exp\left(-\frac{\omega}{2}r^2\right)L_k^{(\ell_F)}(\omega r^2)(\sin\theta)^{\tilde{\ell}+1}(\cos\theta)^{\ell+1}P_n^{\tilde{\ell}+1/2,\ell+1/2}(\cos 2\theta). \tag{31}$$

The quantum numbers $\{k, n\}$ denote the azimuthal quantum number and the number of nodes of the wavefunction, respectively. Both are positive.

The wavefunctions are orthogonal for $\theta \in [0, \pi/2]$ in the sense that

$$\int_0^{+\infty} r\,\mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta\, \Phi_{k,n}(r,\theta)\Phi_{k',n'}(r,\theta) = \delta_{k,k'}\delta_{n,n'}N_{k,n} \tag{32}$$

with

$$N_{k,n} = \frac{\Gamma(\ell_F+1+k)\Gamma(3/2+\ell+n)\Gamma(3/2+\tilde{\ell}+n)}{4k!n!(2+\ell+\tilde{\ell}+2n)\Gamma(\ell+\tilde{\ell}+2+n)}\omega^{-\ell_F-1}. \tag{33}$$

The separability of the variables and the fact that the solutions are expressed by Laguerre and Jacobi polynomials ensures that the solution constitutes an orthonormal basis.

Expressed in terms of R, s variables, the eigensolutions read

$$\Psi_{n,k}^{(\ell,\ell')}(R,s) = (R^2+s^2)^{(\ell_F-\tilde{\ell}-\ell-2)/2}\exp\left(-\frac{\omega}{2}(R^2+s^2)\right)L_k^{(\ell_F)}(\omega(R^2+s^2))$$
$$\times s^{\tilde{\ell}+1}R^{\ell+1}P_n^{\tilde{\ell}+1/2,\ell+1/2}\left(\frac{R^2-s^2}{R^2+s^2}\right). \tag{34}$$

In the absence of interaction between particles 1 and 2, i.e. $\lambda = 0$, we have $\tilde{\ell} = \ell'$ and $\ell_F = \ell + \ell' + 2(n+1)$. It leads to

$$\Psi_{n,k}^{(\ell,\ell')}(R,s) = (R^2+s^2)^n\exp\left(-\frac{\omega}{2}(R^2+s^2)\right)L_k^{(\ell+\ell'+2(n+1))}(\omega(R^2+s^2))$$
$$\times s^{\ell'+1}R^{\ell+1}P_n^{\ell'+1/2,\ell+1/2}\left(\frac{R^2-s^2}{R^2+s^2}\right). \tag{35}$$

The corresponding eigenvalues are trivially given by

$$E_{k,n}(\ell,\ell') = 2\omega(2k+2n+\ell+\ell'+3). \tag{36}$$

3. The model in $D = 2$ dimensions

Before trying to generalize the model to the D dimensional space, we study the $D = 2$ case. Much of what has been derived in the preceding section remains valid; the corresponding Laplacian operators and vectors take their expressions in $D = 2$. In particular, equations (1), (3) and (2) are formally the same.

Introducing the centre of mass and relative coordinates, and by using the circular harmonics

$$\frac{\exp(\mathrm{i}\ell\alpha)}{\sqrt{2\pi}} \qquad \text{with} \qquad \ell \in \mathcal{Z}\ \alpha \in [0, 2\pi], \tag{37}$$

the ansatz (4) takes the form

$$\psi(\vec{R},\vec{s}) = \frac{1}{2\pi}\frac{\psi^{(\ell,\ell')}(R,s)}{\sqrt{Rs}}\exp(\mathrm{i}\ell\alpha_R)\exp(\mathrm{i}\ell'\alpha_s). \tag{38}$$

We obtain the equivalent to equation (5):

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial s^2} - \frac{\ell^2-1/4}{R^2} - \frac{\ell'^2-1/4}{s^2}\right.$$
$$\left. -\omega^2(R^2+s^2) - 2\lambda\frac{R^2-s^2}{(R^2+s^2)s^2} + E\right)\psi^{(\ell,\ell')}(R,s) = 0. \tag{39}$$

Moving to polar coordinates and remembering that $r \geqslant 0$ with $\theta \in [0, \pi/2]$, and setting $\psi^{(\ell,\ell')}(R, s) = \Phi(r, \theta)$ for fixed ℓ and ℓ', equation (39) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\ell^2 - 1/4}{r^2(\cos\theta)^2} - \frac{\ell'^2 - 1/4}{r^2(\sin\theta)^2} \right.$$
$$\left. - \omega^2 r^2 - \frac{2\lambda}{r^2}\frac{(\cos\theta)^2 - (\sin\theta)^2}{(\sin\theta)^2} + E\right)\Phi(r, \theta) = 0. \tag{40}$$

Again, we look for eigenenergies $E_{k,n}$ and eigensolutions $\Phi_{k,n}$, where k and n label the radial and angular states, respectively. Assuming further

$$\Phi_{k,n}(r, \theta) = \frac{v_k(r)}{\sqrt{r}}\phi_n(\theta), \tag{41}$$

we end up with

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\ell^2 - 1/4}{(\cos\theta)^2} - \frac{\ell'^2 - 1/4}{(\sin\theta)^2} - 2\lambda\frac{(\cos\theta)^2 - (\sin\theta)^2}{(\sin\theta)^2} + C_n\right)\phi_n(\theta) = 0 \tag{42}$$

and

$$\left[\frac{\partial^2}{\partial r^2} - \omega^2 r^2 + E_{k,n} - \frac{C_n}{r^2}\right]v_k(r) = 0. \tag{43}$$

As before and under the same conditions, the angular equation

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\ell^2 - 1/4}{(\cos\theta)^2} - \frac{\ell'^2 - 1/4 + 2\lambda}{(\sin\theta)^2} + 4\lambda + C_n\right)\phi_n(\theta) = 0 \tag{44}$$

is solved on $[0, \pi/2]$. Again the singularity of $(\sin\theta)^2$ is analogous to that of a centrifugal barrier. It can be treated if and only if $\ell'^2 + 2\lambda \geqslant 0\,\forall\,\ell'$, which implies $\lambda \geqslant 0$. The centrifugal barrier $(\ell^2 - 1/4)/(\cos\theta)^2$ does not raise difficulties for non-zero ℓ in the vicinity of $\pi/2$. Several self-adjoint extensions exist for $\ell = 0$, and $\ell'^2 + 2\lambda = 0$. In this latter case we consider the Friedrich extension of the operator.

We define $\tilde{\ell}$ by

$$\tilde{\ell}^2 = \ell'^2 + 2\lambda \tag{45}$$

and take the positive root

$$\tilde{\ell} = \sqrt{\ell'^2 + 2\lambda}, \tag{46}$$

so that $\tilde{\ell} = \ell'$ for $\lambda = 0$. Equation (44) becomes

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{\ell^2 - 1/4}{(\cos\theta)^2} - \frac{\tilde{\ell}^2 - 1/4}{(\sin\theta)^2} + 4\lambda + C_n\right)\phi_n(\theta) = 0. \tag{47}$$

The ansatz

$$\phi_n(\theta) = (\sin\theta)^{\tilde{\ell}+1/2}(\cos\theta)^{\ell+1/2}g_n(y) \qquad y = \cos 2\theta \tag{48}$$

transforms (47) to

$$\left[(1 - y^2)\frac{\partial^2}{\partial y^2} + (\ell - \tilde{\ell} - (2 + \ell + \tilde{\ell})y\frac{\partial}{\partial y} + \frac{C_n}{4} + \lambda - \frac{(\ell + \tilde{\ell} + 1)^2}{4}\right]g_n(y) = 0, \tag{49}$$

where

$$\lambda = \frac{(\tilde{\ell} - \ell')(\tilde{\ell} + \ell')}{2}. \tag{50}$$

The general solutions in terms of Jacobi polynomials $P_n(\tilde{\ell}, \ell, y)$ imply

$$C_n = -4\lambda + (\ell + \tilde{\ell} + 1)^2 + 4n(n + \ell + \tilde{\ell} + 1). \tag{51}$$

The general solution of (44) then reads

$$\phi_n(\theta) = (\sin\theta)^{\tilde{\ell}+1/2}(\cos\theta)^{\ell+1/2} P_n^{\tilde{\ell},\ell}(\cos 2\theta). \tag{52}$$

As far as the radial equation (43) is concerned, the constant C_n must be strictly positive $\forall n$ in order to ensure the operator to be self-adjoint. This condition is satisfied if

$$-4\lambda + (\ell + \tilde{\ell} + 1)^2 > 0. \tag{53}$$

Remembering the definition of $\tilde{\ell}$, equation (46), for $\ell \geqslant 0$, the minimal value of $\tilde{\ell}$ (labelled $\tilde{\ell}_c$) is given by

$$\tilde{\ell}_c = \sqrt{2\lambda}. \tag{54}$$

Consequently, the constant $C_n > 0 \, \forall \, n, \ell, \ell'$ if

$$-4\lambda + (\tilde{\ell}_c + 1)^2 > 0, \tag{55}$$

which is verified for

$$0 \leqslant \lambda < \frac{3 + 2\sqrt{2}}{2}. \tag{56}$$

It has to be noted that the acceptable values of λ in $D = 2$ are automatically acceptable in $D = 3$. However, the reciprocal is not true.

Defining ℓ_F by

$$C_n = \ell_F^2 \qquad \ell_F = \sqrt{(\ell + \tilde{\ell} + 2n + 1)^2 - 4\lambda}, \tag{57}$$

the solutions of equation (43) are

$$v_k(r) = r^{\ell_F + 1/2} \exp(-\omega r^2/2) L_k^{(\ell_F)}(\omega r^2). \tag{58}$$

The eigenvalues are given by

$$E_{k,n} = 2\omega(2k + 1 + \ell_F). \tag{59}$$

The final expressions for the wavefunction as the spectrum read

$$\Phi_{k,n}(r,\theta) = r^{\ell_F} \exp\left(-\frac{\omega}{2} r^2\right) L_k^{(\ell_F)}(\omega r^2)(\sin\theta)^{\tilde{\ell}+1/2}(\cos\theta)^{\ell+1/2} P_n^{\tilde{\ell},\ell}(\cos 2\theta) \tag{60}$$

and

$$E_{k,n}(\ell,\tilde{\ell}) = 2\omega\left(2k + 1 + \sqrt{(\ell + \tilde{\ell} + 2n + 1)^2 - 4\lambda}\right). \tag{61}$$

These wavefunctions are also orthogonal for $\theta \in [0, \pi/2]$ according to equations (32) and (33) where in (33) and in the expressions of $\tilde{\ell}$ and ℓ_F we substitute $\ell + 1/2$ and $\ell' + 1/2$ by ℓ and ℓ', respectively.

Expressed in terms of R, s variables, the eigensolutions read

$$\Psi_{n,k}^{(\ell,\ell')}(R,s) = (R^2 + s^2)^{(\ell_F - \tilde{\ell} - \ell - 1)/2} \exp\left(-\frac{\omega}{2}(R^2 + s^2)\right) L_k^{(\ell_F)}(\omega(R^2 + s^2))$$
$$\times s^{\tilde{\ell}+1/2} R^{\ell+1/2} P_n^{\tilde{\ell},\ell}\left(\frac{R^2 - s^2}{R^2 + s^2}\right). \tag{62}$$

In the absence of interaction , we have $\lambda = 0$, $\tilde{\ell} = \ell'$ and $\ell_F = \ell + \ell' + 2n + 1$. It leads to

$$\Psi_{n,k}^{(\ell,\ell')}(R,s) = (R^2 + s^2)^n \exp\left(-\frac{\omega}{2}(R^2 + s^2)\right) L_k^{(\ell+\ell'+2n+1)}(\omega(R^2 + s^2))$$
$$\times s^{\ell'+1/2} R^{\ell+1/2} P_n^{\ell',\ell}\left(\frac{R^2 - s^2}{R^2 + s^2}\right). \tag{63}$$

4. The model in D dimensions

The generalization to the D-dimensional space follows the same strategy as in sections 2 and 3. The centre of mass and relative coordinates, together with the use of the hyperspherical harmonics [11], allow us to write

$$\psi(\vec{R},\vec{s}) = \frac{\psi^{(\ell,\ell')}(R,s)}{(Rs)^{(D-1)/2}} Y_{\ell,[M]}(\Omega_R) Y_{\ell',[M']}(\Omega_s). \tag{64}$$

Here $[M]$ denotes the set $[M] = \{m_1, m_2, \ldots, m_p\}$, $p = D-2$ satisfying $\ell = m_0 \geqslant m_1 \geqslant m_2 \cdots \geqslant m_p \geqslant 0$, and [11]

$$Y_{\ell,[M]} = \mathrm{e}^{\pm \mathrm{i} m_p \phi} \prod_{k=1}^{p} (\sin\theta_k)^{m_k} \prod_{k=0}^{p-1} C_{m_k - m_{k+1}}^{m_{k+1}+p/2-k/2}(\cos\theta_{k+1}). \tag{65}$$

The hyperspherical harmonics $Y_{\ell,[M]}$ are given in terms of the Gegenbauer polynomials $C_n^a(x)$. We recall that the hyperspherical polar coordinates are given by

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\cdots \\
x_{p+1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_p\cos\phi \\
x_{p+2} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_p\sin\phi
\end{aligned} \tag{66}$$

with $\theta_k \in [0,\pi]$, $k = 1, 2, \ldots, p$, and $\phi \in [0, 2\pi]$.

Setting $(D-3)/2 = m$ (here $m \geqslant 0$), the equivalent to equation (5) reads

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial s^2} - \frac{(\ell+m)(\ell+m+1)}{R^2} - \frac{(\ell'+m)(\ell'+m+1)}{s^2} - \omega^2(R^2+s^2) - 2\lambda\frac{R^2-s^2}{(R^2+s^2)s^2} + E\right)\psi^{(\ell,\ell')}(R,s) = 0. \tag{67}$$

In polar coordinates (6), with $r \geqslant 0$ and $\theta \in [0,\pi/2]$, together with $\psi^{(\ell,\ell')}(R,s) = \Phi(r,\theta)$, we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{(\ell+m)(\ell+m+1)}{r^2(\cos\theta)^2} - \frac{(\ell'+m)(\ell'+m+1)}{r^2(\sin\theta)^2} - \omega^2 r^2 - \frac{2\lambda}{r^2}\frac{(\cos\theta)^2-(\sin\theta)^2}{(\sin\theta)^2} + E\right)\Phi(r,\theta) = 0. \tag{68}$$

We look for eigenenergies $E_{k,n}$ and eigensolutions $\Phi_{k,n}$, where k and n label the radial and angular states, respectively. Assuming

$$\Phi_{k,n}(r,\theta) = \frac{v_k(r)}{\sqrt{r}}\phi_n(\theta), \tag{69}$$

we obtain the two differential equations

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{(\ell+m)(\ell+m+1)}{(\cos\theta)^2} - \frac{(\ell'+m)(\ell'+m+1)}{(\sin\theta)^2} - 2\lambda\frac{(\cos\theta)^2-(\sin\theta)^2}{(\sin\theta)^2} + C_n\right)\phi_n(\theta) = 0 \tag{70}$$

and

$$\left[\frac{\partial^2}{\partial r^2} - \omega^2 r^2 + E_{k,n} - \frac{C_n}{r^2}\right] v_k(r) = 0. \tag{71}$$

The angular equation is solved on $[0, \pi/2]$. It is rewritten as

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{(\ell+m)(\ell+m+1)}{(\cos\theta)^2} - \frac{(\ell'+m)(\ell'+m+1)+2\lambda}{(\sin\theta)^2} + 4\lambda + C_n\right)\phi_n(\theta) = 0. \tag{72}$$

The centrifugal barrier in the vicinity of $\theta = 0$ can be treated for every integer ℓ' if and only if $1/4 + 2\lambda + m(m+1) > 0$ which is equivalent to $2\lambda + D^2/4 - D + 1 > 0$.

Here $\tilde{\ell}$ is defined by

$$\tilde{\ell} = -m - \tfrac{1}{2} + \sqrt{(\ell'+m+1/2)^2 + 2\lambda}, \tag{73}$$

where we have considered the positive root of (13) in order to recover $\tilde{\ell} = \ell'$ for $\lambda = 0$. Equation (72) becomes

$$\left(\frac{\partial^2}{\partial\theta^2} - \frac{(\ell+m)(\ell+m+1)}{(\cos\theta)^2} - \frac{(\tilde{\ell}+m)(\tilde{\ell}+m+1)}{(\sin\theta)^2} + 4\lambda + C_n\right)\phi_n(\theta) = 0. \tag{74}$$

By using the same kind of ansatz than in the preceding sections, we set

$$\begin{aligned}\phi_n(\theta) &= (\sin\theta)^{\tilde{\ell}+m+1}(\cos\theta)^{\ell+m+1} g_n(y)\\ y &= \cos 2\theta.\end{aligned} \tag{75}$$

It transforms (74) to

$$\begin{aligned}\Bigg[(1-y^2)\frac{\partial^2}{\partial y^2} + (\ell - \tilde{\ell} - (3+2m+\ell+\tilde{\ell})y\frac{\partial}{\partial y} + \frac{C_n}{4} + \lambda\\ - \frac{(\ell+\tilde{\ell}+2+2m)^2}{4}\Bigg] g_n(y) = 0,\end{aligned} \tag{76}$$

where (see equation (73))

$$\lambda = \frac{(\tilde{\ell}-\ell')(\tilde{\ell}+\ell'+1+2m)}{2}. \tag{77}$$

The general solutions are given by Jacobi polynomials $P_n(\tilde{\ell}+m+1/2, \ell+m+1/2, y)$ provided that C_n fulfils the condition

$$C_n = -4\lambda + (\ell+\tilde{\ell}+2m+2)^2 + 4n(n+\ell+\tilde{\ell}+2m+2). \tag{78}$$

The constant C_n has to be strictly positive to ensure the operator of the radial equation to be self-adjoint. Several self-adjoint extensions occur at $C_n = 0$. The inequality for C_n has to be satisfied for n, ℓ, ℓ' non-negative integer, which implies $\lambda < \lambda_c$. It defines the acceptable domain of λ:

$$-\frac{(2m+1)^2}{8} \leqslant \lambda < \frac{7+10m+4m^2+(3+2m)\sqrt{5+8m+4m^2}}{2}. \tag{79}$$

Remembering $m = (D-3)/2$, this general expression shows that the domain of λ leading to a solvable model increases monotonically as the dimension of the space increases.

The auxiliary quantity ℓ_F

$$C_n = \ell_F^2 \qquad \ell_F = \sqrt{(\ell+\tilde{\ell}+2n+2m+2)^2 - 4\lambda} \tag{80}$$

allows us to write the radial solutions as

$$v_k(r) = r^{\ell_F+1/2}\exp(-\omega r^2/2)L_k^{(\ell_F)}(\omega r^2). \tag{81}$$

The eigenvalues are given by

$$E_{k,n} = 2\omega(2k + \ell_F + 1). \tag{82}$$

Finally the full wavefunctions take the form

$$\Phi_{k,n}(r,\theta) = r^{\ell_F} \exp\left(-\frac{\omega}{2}r^2\right) L_k^{(\ell_F)}(\omega r^2)(\sin\theta)^{\tilde{\ell}+m+1}(\cos\theta)^{\ell+m+1} P_n^{\tilde{\ell}+m+1/2,\ell+m+1/2}(\cos 2\theta) \tag{83}$$

and the corresponding eigenvalues obey

$$E_{k,n} = 2\omega\left(2k + 1 + \sqrt{(\ell + \tilde{\ell} + 2n + 2m + 2)^2 - 4\lambda}\right). \tag{84}$$

The eigensolutions are orthogonal for $\theta \in [0, \pi/2]$ in the sense given by (32) and the norm is given by equation (33), where ℓ and ℓ' are replaced by $\ell + m$ and $\ell' + m$ including in the definition of $\tilde{\ell}$ and ℓ_F.

5. Supersymmetry

It is simple to examine if the potential equation (74) is supersymmetric and shape invariant. The basic elements used in this section are well known. We refer the reader to the comprehensive paper by Dutt, Khare and Sukhatme [12].

For this purpose we introduce the superpotential

$$W(\theta) = -(\tilde{\ell} + m + 1)\cot\theta + (\ell + m + 1)\tan\theta \tag{85}$$

and consider

$$V_\pm = W^2 \pm \frac{\mathrm{d}}{\mathrm{d}\theta}W. \tag{86}$$

We obtain

$$\begin{aligned} V_+(\theta) &= \frac{(\ell+m+1)(\ell+m+2)}{(\cos\theta)^2} + \frac{(\tilde{\ell}+m+1)(\tilde{\ell}+m+2)}{(\sin\theta)^2} - (2+\ell+\tilde{\ell}+2m)^2 \\ V_-(\theta) &= \frac{(\ell+m)(\ell+m+1)}{(\cos\theta)^2} + \frac{(\tilde{\ell}+m)(\tilde{\ell}+m+1)}{(\sin\theta)^2} - (2+\ell+\tilde{\ell}+2m)^2. \end{aligned} \tag{87}$$

This part of the potential is shape invariant.

As far as the radial part is concerned, it is known to be supersymmetric and shape invariant [12], as shown in [6]

Consequently the D-dimensional model is supersymmetric and shape invariant.

6. $SU(1,1)$ symmetry

In the paper by Meljanac *et al* [7], the resolvability of the model in $D = 1$ was studied in connection with its conformal properties. The question arises of the role of the conformal symmetry in the D-dimensional case.

It is intuitively suspected that the D-dimensional model has the conformal symmetry, because the potential admits the separation of radial and angular variables. Moreover, the potential depends only on the azimuthal angle θ. However, this is not sufficient. To get the correct answer we follow the procedure of Meljanac *et al* [7].

Accordingly, we define

$$T_+ = \frac{1}{2}r^2$$
$$T_- = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - V\right) \tag{88}$$
$$T_0 = \frac{1}{2}\left(r\frac{\partial}{\partial r} + 1\right).$$

From (68), we have

$$V = \frac{(\ell+m)(\ell+m+1)}{r^2\cos(\theta)^2} + \frac{(\ell'+m)(\ell'+m+1)}{r^2\sin(\theta)^2} + \frac{2\lambda}{r^2}\frac{\cos(\theta)^2-\sin(\theta)^2}{\sin(\theta)^2}. \tag{89}$$

The commutation relations

$$[T_0, T_+] = T_+ \qquad [T_-, T_+] = 2T_0 \tag{90}$$

are verified whatever $V(r,\theta)$ is. In contrast,

$$[T_0, T_-] = -T_- \tag{91}$$

is satisfied if and only if

$$2V(r,\theta) + r\frac{\partial}{\partial r}V(r,\theta) = 0. \tag{92}$$

One integrated, this is equivalent to

$$V(r,\theta) = \frac{C(\theta)}{r^2}. \tag{93}$$

Here C is a function of θ.

From (68), we see that our Hamiltonian in the D-dimensional case belongs to the class of conformaly invariant systems. The requirement (92) for the potential to be a real homogeneous function of order -2 has already been noticed by Meljanac *et al* [7].

7. Conclusions

We have generalized to the D-dimensional space a model of the Calogero type developed in $D = 1$ in a recent work [6]. The model describes two interacting particles in a harmonic mean field generated by an infinitely heavy third particle. For given domains of the coupling constant, the model is exactly solvable in D dimensions. The domain of validity is found to increase with D.

We show further that in all cases the potentials for the angular and radial equations are shape invariant. The whole model is supersymmetric. This property could be the reason for the exact resolvability of the model in the D-dimensional space. A more formal reason is that the D-dimensional Hamiltonian belongs to the class of conformaly invariant systems.

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